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An extension of gap functions for a system of vector equilibrium problems with applications to optimization problems

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Abstract In this paper, the notion of gap functions is extended from scalar case to vector one. Then, gap functions and generalized functions for several kinds of vector equilibrium problems are shown. As an application, the dual problem of a class of optimization problems with a system of vector equilibrium constraints (in short, OP) is established, the concavity of the dual function, the weak duality of (OP) and the saddle point sufficient condition are derived by using generalized gap functions.

Keywords Vector equilibrium problem · Optimization problem · Generalized gap function · The Lagrangian function · Saddle point · Weak duality

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1 Introduction

Recently, the vector equilibrium problem (in short, VEP) with fixed or moving cones in abstract spaces has been studied intensively by many authors (see, for example, [1–4,6,8,10,11,13,14,17,19,20,22,28,29] and the references therein). (VEP) contains as special cases, for instance, vector variational inequality and vector complementarity problems, as well as vector optimization problems (see [5,8,12,16,17,19,21,25,30]).

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The concept of a gap function was first introduced for the study of a convex optimization problem. The meaning of "gap" is interpreted as the difference between the cost function and the restricted Wolfe dual. The concept of gap functions was then applied to variational inequality problems. As is well known, gap functions play a crucial role in transforming a variational inequality problem into an optimization problem [18,24]. Thus, powerful optimization solution methods and algorithms can be applied for finding solutions of variational inequalities. In [7], Chen, Goh and Yang extended the theory of gap functions for scalar variational inequality problems to the case of vector variational inequality problems. In 2002, Yang and Yao [33] introduced gap functions for vector variational inequalities with point-to-set mappings. In 2003, the gap function approach was extended to the study for scalar equilibrium problem (in short, EP) by Mastroeni [31]. Some related works, we refer to [26,27,32]. By virtue of the nonlinear scalarization function introduced by Chen, Yang and Yu [10], the gap function for a system of equilibrium problems (in short, SVEP) was introduced by Huang, Li and Yao [23] and the necessary and sufficient conditions for (SVEP) were established.

Inspired and motivated by above research works, in this paper, we extend the notion of gap functions from scalar case to vector one. Then, we show gap functions and generalized functions for several kinds of vector equilibrium problems. As an application, we establish the dual problem of a class of optimization problems with a system of vector equilibrium constraints (in short, OP), derive the concavity of the dual function, the weak duality of (OP) and the saddle point sufficient condition by using generalized gap functions under certain conditions.

2 Preliminaries

Throughout this paper, without other specifications, let $I = \{1, 2, \ldots, n\}$ be the index set, and for each $i \in I$, let X_i and Y_i be locally convex Hausdorff spaces. Consider a family of nonempty closed convex subsets $\{K_i\}_{i\in I}$ with K_i in X_i. We denote by $X = \prod_{i\in I} X_i$, $Y = \prod_{i\in I} Y_i$ and $K = \prod_{i\in I} K_i$. For each $i \in I$, let $C_i : K \to 2^{Y_i}$ be a point-to-set mapping such that, for any $x \in K$, $C_i(x)$ is a pointed, closed and convex cone in Y_i with nonempty interior $\text{int}C_i(x)$. For each $i \in I$, let $e_i : K \to Y_i$ be a vector-valued mapping, and for any $x \in K, e_i(x) \in \text{int}C_i(x)$. For each $i \in I$, let $f_i : K \times K_i \to Y_i$ be a bifunction. We consider the following system of vector equilibrium problems (in short, SVEP) which is to find $x^* \in K$ such that for each $i \in I$,

$$\text{(SVEP)} \qquad f_i(x^*, y_i) \notin -\text{int}C_i(x^*), \quad \forall y_i \in K_i.$$

In the case that I is a countable (or an uncountable) index set, (SVEP) has been studied by Ansari, Chan and Yang [1], Huang, Li and Yao [23].

If the index set I is a singleton, then (SVEP) collapses to the following generalized vector equilibrium problems (in short, $(\text{GVEP})^1$): finding $x^* \in K$ such that

$$(\text{GVEP})^1 \qquad f(x^*, y) \notin -\text{int}C(x^*), \quad \forall y \in K,$$

where $f : K \times K \to Y$ is a bifunction, $C : K \to 2^Y$ is a point-to-set mapping such that, for any $x \in K$, $C(x)$ is a pointed, closed and convex cone in Y with nonempty interior $\text{int}C(x)$, and $e : K \to Y$ is a vector-valued mapping such that, for any $x \in K, e(x) \in \text{int}C(x)$.

If the index set I is a singleton, $Y = R^n$ and $C(x) = R^n_+ = \{x = (x_1, \ldots, x_n)^T : x_l \geq 0, l = 1, \ldots, n\}$ for all $x \in K$, where the superscript T denotes the transpose, then $(\mathrm{GVEP})^1$ reduces to the following generalized vector equilibrium problems (in short, GVEP): finding $x^* \in K$ such that

$$(\mathrm{GVEP}) \qquad f(x^*, y) \notin -\mathrm{int}R^n_+, \quad \forall y \in K,$$

or equivalently,

$$(\mathrm{GVEP}) \quad \forall y \in K, \quad \exists l_y \in I, \quad \text{such that} \quad f_{l_y}(x^*, y) \geq 0,$$

where $f_{l_y}(x^*, y)$ denotes the l_yth component of $f(x^*, y)$.

Let $f : K \times K \to R^n$ be a bifunction, we also consider the following vector equilibrium problems (in short, VEP): finding $x^* \in K$ such that

$$(\mathrm{VEP}) \qquad f(x^*, y) \in R^n_+, \quad \forall y \in K,$$

or equivalently,

$$(\mathrm{VEP}) \quad \forall l \in I \quad \text{such that} \quad f_l(x^*, y) \geq 0, \quad \forall y \in K,$$

where $f_l(x^*, y)$ denotes the lth component of $f(x^*, y)$.

If the index set I is a singleton, $Y = R$ and $C(x) = R_+ = [0, +\infty)$ for all $x \in K$, then (SVEP) reduces to the classical equilibrium problems (in short, EP): finding $x^* \in K$ such that

$$(\mathrm{EP}) \qquad f(x^*, y) \geq 0, \quad \forall y \in K.$$

We denote by E^S, E^{G^1}, E^G, E^0 and E the solution sets of (SVEP), $(\mathrm{GVEP})^1$, (GVEP), (VEP) and (EP), respectively. It is clear that $E^0 \subseteq E^G$.

Definition 2.1 A function $p : K \to R$ is said to be a gap function for (SVEP) (or $(\mathrm{GVEP})^1$, (GVEP), (VEP), (EP)) if it satisfies the following properties:

(i) $p(x) \leq 0$ for all $x \in K$;
(ii) $p(x^*) = 0$ if and only if $x^* \in E^S$ (or E^{G^1}, E^G, E^0, E).

In [7], Chen, Goh and Yang defined and derived some point-to-set mappings as gap functions for vector variational inequalities. Now, we define vector functions as gap functions as follows:

Definition 2.2 A function $q : K \to R^n$ is said to be a generalized gap function for (SVEP) (or (VEP)) if it satisfies the following properties:

(i) $q(x) \in -R^n_+$ for all $x \in K$;
(ii) $q(x^*) = 0$ if and only if $x^* \in E^S$(or E^0).

In 1990, Gerth and Weidner first derived the nonconvex separation theorems for any arbitrary set and any not necessarily convex set in a topological vector space. The following nonlinear scalarizing function derived by Chen and Yang [9], Chen, Yang and Yu [10] is a generalization of the scalarizing function introduced by Gerth and Weidner [15].

Definition 2.3 [10] Let X and Y be two locally convex Hausdorff topological vector spaces, $C : X \to 2^Y$ a point-to-set mapping such that for any $x \in X$, $C(x)$ is a proper, pointed, closed and convex cone in Y with $\mathrm{int}C(x) \neq \emptyset$. Let $e : X \to Y$ be a vector-valued mapping, and for any $x \in X, e(x) \in \text{int}C(x)$. The nonlinear scalarization function $\xi_e : X \times Y \to R$ is defined as follows:

$$\xi_e(x,y) \stackrel{\textbf{def}}{=} \inf\{\lambda \in R : y \in \lambda e(x) - C(x)\}, \quad \forall (x,y) \in X \times Y.$$

If $e(x) \equiv k^0$ for all $x \in X$, then the nonlinear scalarization function ξ_e reduces to the nonlinear scalarization function ξ_{k^0} introduced by Chen and Yang [9]. Furthermore, if $e(x) \equiv k^0$ and $C(x) \equiv C$ for all $x \in X$, then the nonlinear scalarization function ξ_e collapses to the nonlinear scalarization function ξ_{k^0} introduced by Gerth and Weidner [15].

The following results are very important properties of the nonlinear scalarization function ξ_e.

Lemma 2.1 *[10,15] Let X and Y be two locally convex Hausdorff topological vector spaces, $C : X \to 2^Y$ a point-to-set mapping such that for any $x \in X$, $C(x)$ is a proper, pointed, closed and convex cone in Y with int$C(x) \neq \emptyset$. Let $e : X \to Y$ be a vector-valued mapping, and for any $x \in X, e(x) \in \text{int}C(x)$. For each $\lambda \in R$ and $(x,y) \in X \times Y$, we have*

(i) $\xi_e(x,y) < \lambda \Leftrightarrow y \in \lambda e(x) - \text{int}C(x)$;
(ii) $\xi_e(x,y) \leq \lambda \Leftrightarrow y \in \lambda e(x) - C(x)$;
(iii) $\xi_e(x,y) \geq \lambda \Leftrightarrow y \notin \lambda e(x) - \text{int}C(x)$;
(iv) $\xi_e(x,y) > \lambda \Leftrightarrow y \notin \lambda e(x) - C(x)$;
(v) $\xi_e(x,y) = \lambda \Leftrightarrow y \in \lambda e(x) - \partial C(x)$,

where $\partial C(x)$ is topological boundary of $C(x)$.

3 Gap functions and generalized gap functions for several kinds of vector equilibrium problems

In this section, we give gap functions and the generalized gap functions for (SVEP), $(\text{GVEP})^1$, (GVEP) and (VEP), and show the equivalence of solutions for them by using gap functions and generalized gap functions.

3.1 Gap functions for (SVEP), $(\text{GVEP})^1$, (GVEP) and (VEP)

We first consider the gap functions for (SVEP), $(\text{GVEP})^1$, (GVEP) and (VEP).

In [23], Huang, Li and Yao proved the following theorem:

Theorem 3.1 [23] *If for any $x \in K$ and each $i \in I$, $f_i(x,x_i) \in -\partial C_i(x)$, where x_i is the ith component of x, then the function $\phi(x)$ is a gap function for (SVEP), where the function $\phi : K \to R$ is defined as follows:*

$$\phi(x) = \inf_{i \in I}\{-\phi_0(x,i)\}$$

and

$$\phi_0(x,i) = \sup_{y_i \in K_i}\{-\xi_{e_i}(x, f_i(x,y_i))\}.$$

Moreover, $E^S = \{x \in K : \phi(x) = 0\}$.

Corollary 3.1 *Suppose that the index set I is a singleton, $f : K \times K \to Y$ is a bifunction, $C : K \to 2^Y$ is a point-to-set mapping such that, for any $x \in K$, $C(x)$ is a pointed, closed and convex cone in Y with nonempty interior int$C(x)$, and $e : K \to Y$ is a vector-valued mapping such that, for any $x \in K, e(x) \in \text{int}C(x)$. If for any $x \in K$, $f(x,x) \in -\partial C(x)$, then the function $\phi(x)$ is a gap function for $(GVEP)^1$, where the function $\phi : K \to R$ is defined as follows:*

$$\phi(x) = -\sup_{y \in K}\{-\xi_e(x, f(x,y))\}.$$

In addition, $E^{G^1} = \{x \in K : \phi(x) = 0\}$.

Proof The proof is the same as in Theorem 3.1, and so we omit it.
From Corollary 3.1, we obtain the following conclusion. □

Corollary 3.2 *Suppose that the index set I is a singleton, $f : K \times K \to R^n$ is a bifunction, $C(x) = R^n_+ = \{x = (x_1, \ldots, x_n)^T : x_l \geq 0, l = 1, \ldots, n\}$ and $e(x) = (1, \ldots, 1)^T$ for all $x \in K$. If for any $x \in K$, $f(x,x) \in -\partial R^n_+$, then the function $\phi(x)$ is a gap function for $(GVEP)$, where the function $\phi : K \to R$ is defined as follows:*

$$\phi(x) = -\sup_{y \in K}\{-\xi_e(x, f(x,y))\}.$$

Moreover, $E^G = \{x \in K : \phi(x) = 0\}$.

Remark that the gap function for $(GVEP)$ in Corollary 3.2 is defined by means of the nonlinear scalarization function ξ_e. Now, we derive another gap function for $(GVEP)$ without the help of the function ξ_e.

Theorem 3.2 *Suppose that the index set I is a singleton, $f : K \times K \to R^n$ is a bifunction, and $C(x) = R^n_+ = \{x = (x_1, \ldots, x_n)^T : x_l \geq 0, l = 1, \ldots, n\}$ for all $x \in K$. If for any $x \in K$, $f(x,x) \in -R^n_+$, then the function $\phi(x)$ is a gap function for $(GVEP)$, where the function $\phi : K \to R$ is defined as follows:*

$$\phi(x) = -\sup_{y \in K}\{\phi_0(x,y)\}$$

and

$$\phi_0(x,y) = \min_{l \in I}\{-f_l(x,y)\},$$

where $f_l(x,y)$ is the lth component of $f(x,y)$. Furthermore, $E^G = \{x \in K : \phi(x) = 0\}$.

Proof We first show that $\phi(x)$ is a gap function for (GVEP). The proof consists of two steps.

(i) Since for any $x \in K, f(x,x) \in -R^n_+$,

$$f_l(x,x) \leq 0, \quad l = 1, \ldots, n.$$

It follows that

$$\phi_0(x,x) = \min_{l \in I}\{-f_l(x,x)\} \geq 0$$

and hence

$$\phi(x) = -\sup_{y \in K}\{\phi_0(x,y))\} \leq -\phi_0(x,x) \leq 0, \qquad \forall x \in K.$$

(ii) If $\phi(x^*) = 0$, then $x^* \in K$ and

$$-\sup_{y\in K}\{\min_{l\in I}\{-f_l(x^*, y)\}\} = 0.$$

Then, for any $y \in K$,

$$\min_{l\in I}\{-f_l(x^*, y)\} \leq 0,$$

which implies that there exists $l_y \in I$ such that $-f_{l_y}(x^*, y) \leq 0$. Thus, $f(x^*, y) \notin -\text{int}R^n_+$ for all $y \in K$ and so $x^* \in E^G$.

Conversely, if $x^* \in E^G$, then $x^* \in K$ and

$$f(x^*, y) \notin -\text{int}R^n_+, \quad \forall y \in K.$$

Then, for any $y \in K$, there exists $l_y \in I$ such that $f_{l_y}(x^*, y) \geq 0$. It follows that

$$\phi_0(x^*, y) = \min_{l\in I}\{-f_l(x^*, y)\} \leq -f_{l_y}(x^*, y) \leq 0$$

and so

$$\phi(x^*) = -\sup_{y\in K}\{\phi_0(x^*, y)\} \geq 0. \tag{3.1}$$

Now, (i) and inequality (3.1) imply that $\phi(x^*) = 0$.

It follows directly from the proof of (ii) that $E^G = \{x \in K : \phi(x) = 0\}$ holds and thus the proof is complete. □

Remark 3.1 Suppose that the index set I is a singleton, $f : K\times K \to R^n$ is a bifunction, $C(x) = R^n_+ = \{x = (x_1, \ldots, x_n)^T : x_l \geq 0, l = 1, \ldots, n\}$ and $e(x) = (1, \ldots, 1)^T$ for all $x \in K$ for all $x \in K$. If for any $x \in K$, $f(x,x) \in -\partial R^n_+$, then the gap function $\phi(x)$ defined in Corollary 3.2 by means of the nonlinear scalarization function ξ_e is same as one in Theorem 3.2 without the help of the function ξ_e. That is, for any $x \in K$,

$$-\sup_{y\in K}\{\min_{l\in I}\{-f_l(x, y)\}\} = -\sup_{y\in K}\{-\xi_e(x, f(x, y))\}.$$

In fact, since

$$-\sup_{y\in K}\{\min_{l\in I}\{-f_l(x, y)\}\} = -\sup_{y\in K}\{-\max_{l\in I} f_l(x, y)\},$$

it suffices to show

$$\max_{l\in I} f_l(x, y) = \xi_e(x, f(x, y)), \quad \forall (x, y) \in K \times K.$$

From the definition of ξ_e, one has, for any $(x, y) \in K \times K$,

$$\begin{aligned}
\xi_e(x, f(x, y)) &= \inf\{\lambda \in R : f(x, y) \in \lambda(1, \ldots, 1)^T - R^n_+\} \\
&= \inf\{\lambda \in R : (f_1(x, y) - \lambda, f_2(x, y) - \lambda, \ldots, f_n(x, y) - \lambda)^T \in -R^n_+\} \\
&= \inf\{\lambda \in R : f_l(x, y) \leq \lambda, \quad \forall l = 1, 2, \ldots, n\} \\
&= \max_{l\in I} f_l(x, y),
\end{aligned}$$

which yields the desired conclusion.

Theorem 3.3 *Suppose that all conditions in Theorem 3.2 are satisfied. If for any $x \in K$, $f(x,x) \in -R^n_+$, then the function $\phi(x)$ is a gap function for (VEP), where the function $\phi : K \to R$ is defined as follows:*

$$\phi(x) = \min_{l \in I}\{-\phi_0(x,l)\}$$

and

$$\phi_0(x,l) = \sup_{y \in K}\{-f_l(x,y)\},$$

where $f_l(x,y)$ is the lth component of $f(x,y)$. In addition, $E^0 = \{x \in K : \phi(x) = 0\}$.

Proof The proof of $\phi(x)$ is a gap function for (VEP) consists of two steps.

(i) Since for any $x \in K, f(x,x) \in -R^n_+$,

$$f_l(x,x) \le 0, \quad l = 1,2,\dots,n,$$

and so

$$\phi_0(x,l) = \sup_{y \in K}\{-f_l(x,y)\} \ge -f_l(x,x) \ge 0, \quad l = 1,2,\dots,n.$$

Thus

$$\phi(x) = \min_{l \in I}\{-\phi_0(x,l)\} \le 0, \qquad \forall x \in K.$$

(ii) If $\phi(x^*) = 0$, then $x^* \in K$ and

$$\min_{l \in I}\{-\sup_{y \in K}\{-f_l(x^*,y)\}\} = 0.$$

Then, for each $l \in I$, we obtain $-\sup_{y \in K}\{-f_l(x^*,y)\} \ge 0$, which implies that, for any $y \in K$, $-f_l(x^*,y) \le 0$. Thus, we have $f(x^*,y) \in R^n_+$ for all $y \in K$. That is, $x^* \in E^0$.

Conversely, if $x^* \in E^0$, then $x^* \in K$ and

$$f(x^*,y) \in R^n_+, \quad \forall y \in K.$$

That is, for each $l \in I, f_l(x^*,y) \ge 0$ for all $y \in K$. Then,

$$\phi_0(x^*,l) = \sup_{y \in K}\{-f_l(x^*,y)\} \le 0$$

and it follows that

$$\phi(x^*) = \min_{l \in I}\{-\phi_0(x^*,l)\} \ge 0. \tag{3.2}$$

Thus, both (i) and inequality (3.2) yield $\phi(x^*) = 0$.

The relation $E^0 = \{x \in K : \phi(x) = 0\}$ follows directly form the proof of (ii) and thus the proof is complete.

From Theorem 3.3, we have the following result. □

Corollary 3.3 [31] *Suppose that $f : K \times K \to R$ is a bifunction and $C(x) = R_+ = [0,+\infty)$ for all $x \in K$. If for any $x \in K$, $f(x,x) \le 0$, then the function $\phi(x)$ is a gap function for (EP), where the function $\phi : K \to R$ is defined by*

$$\phi(x) = -\sup_{y \in K}\{-f(x,y)\}.$$

Moreover, $E = \{x \in K : \phi(x) = 0\}$.

3.2 Generalized gap functions for (SVEP) and (VEP)

Now, we derive generalized gap functions for (SVEP) and (VEP).

Theorem 3.4 *If for any $x \in K$ and each $i \in I$, $f_i(x, x_i) \in -\partial C_i(x)$, where x_i is the ith component of x, then the function $\varphi(x)$ is a generalized gap function for (SVEP), where the function $\varphi : K \to R^n$ is defined as follows:*

$$\varphi(x) = (-\varphi_1(x), \ldots, -\varphi_n(x))^T$$

and

$$\varphi_i(x) = \sup_{y_i \in K_i} \{-\xi_{e_i}(x, f_i(x, y_i))\}, \quad \forall i = 1, 2, \ldots, n.$$

Moreover, $E^S = \{x \in K : \varphi(x) = 0\}$.

Proof We first prove that the function $\varphi(x)$ is a generalized gap function for (SVEP) which consists of two steps.

(i) Since for any $x \in K$ and each $i \in I$, $f_i(x, x_i) \in -\partial C_i(x)$, from Lemma 2.1 (v), one has

$$\xi_{e_i}(x, f_i(x, x_i)) = 0.$$

It follows that

$$\varphi_i(x) = \sup_{y_i \in K_i} \{-\xi_{e_i}(x, f_i(x, y_i))\} \geq 0,$$

and so

$$\varphi(x) = (-\varphi_1(x), \ldots, -\varphi_n(x))^T \in -R^n_+, \quad \forall x \in K.$$

(ii) If $\varphi(x^*) = 0$, then we obtain

$$(-\sup_{y_1 \in K_1} \{-\xi_{e_1}(x^*, f_1(x^*, y_1))\}, \ldots, -\sup_{y_n \in K_n} \{-\xi_{e_n}(x^*, f_n(x^*, y_n))\})^T = 0.$$

Then $x^* \in K$ and for each $i \in I$,

$$\sup_{y_i \in K_i} \{-\xi_{e_i}(x^*, f_i(x^*, y_i))\} = 0,$$

which implies that, for any $y_i \in K_i$,

$$-\xi_{e_i}(x^*, f_i(x^*, y_i)) \leq 0.$$

From Lemma 2.1 (iii), we conclude $f_i(x^*, y_i) \notin -\text{int}C_i(x^*)$ for all $y_i \in K_i$. That is, $x^* \in E^S$.

Conversely, if $x^* \in E^S$, then $x^* \in K$ and for each $i \in I$,

$$f_i(x^*, y_i) \notin -\text{int}C_i(x^*), \quad \forall y_i \in K_i.$$

By Lemma 2.1 (iii),

$$\xi_{e_i}(x^*, f_i(x^*, y_i)) \geq 0$$

for all $y_i \in K_i$. Then,

$$\varphi_i(x^*) = \sup_{y_i \in K_i} \{-\xi_{e_i}(x^*, f_i(x^*, y_i))\} \leq 0$$

and hence

$$\varphi(x^*) = (-\varphi_1(x^*), \ldots, -\varphi_n(x^*))^T \in R^n_+. \tag{3.3}$$

Thus, (i) and (3.3) yield $\varphi(x^*) = 0$.

It is obvious that the proof of (ii) implies $E^S = \{x \in K : \varphi(x) = 0\}$. This completes the proof.

From the proof of Theorem 3.4, it is easy to see that the following conclusion holds. □

Corollary 3.4 *If for any $x \in K$ and each $i \in I$, $f_i(x, x_i) \in -\partial C_i(x)$, where x_i is the ith component of x, then the function $-\varphi_i(x)$ is a gap function for the ith component of $(SVEP)$ (in short, $(SVEP)_i$) which concists of finding $x^* \in K$ such that*

$$f_i(x^*, y_i) \notin -intC_i(x^*), \quad \forall y_i \in K_i,$$

where the function $\varphi_i : K \to R^n$ is defined in Theorem 3.4. Furthermore, $E^{S_i} = \{x \in K : \varphi_i(x) = 0\}$, where E^{S_i} denotes the solution set of $(SVEP)_i$.

Theorem 3.5 *Suppose that all assumptions in Theorem 3.3 are satisfied. If for any $x \in K$, $f(x, x) \in -R^n_+$, then the function $\varphi(x)$ is a generalized gap function for (VEP), where the function $\varphi : K \to R^n$ is defined as follows:*

$$\varphi(x) = (-\varphi_1(x), \ldots, -\varphi_n(x))^T$$

and

$$\varphi_l(x) = \sup_{y \in K}\{-f_l(x, y)\}, \quad \forall l = 1, 2, \ldots, n,$$

where $f_l(x, y)$ is the lth component of $f(x, y)$. Moreover, $E^0 = \{x \in K : \varphi(x) = 0\}$.

Proof We first prove that the function $\varphi(x)$ is a generalized gap function for (VEP) which consists of the following two steps.

(i) Since for any $x \in K, f(x, x) \in -R^n_+$, one has

$$f_l(x, x) \le 0, \quad l = 1, 2, \ldots, n,$$

Then

$$\varphi_l(x) = \sup_{y \in K}\{-f_l(x, y)\} \ge 0, \quad l = 1, 2, \ldots, n,$$

and so

$$\varphi(x) = (-\varphi_1(x), \ldots, -\varphi_n(x))^T \in -R^n_+, \ \forall x \in K.$$

(ii) If $\varphi(x^*) = 0$, then

$$(-\sup_{y \in K}\{-f_1(x^*, y)\}, \ldots, -\sup_{y \in K}\{-f_n(x^*, y)\})^T = 0.$$

It follows that, $x^* \in K$ and for each $l \in I$,

$$\sup_{y \in K}\{-f_l(x^*, y)\} = 0,$$

which yields that, for any $y \in K$, $-f_l(x^*, y) \le 0$. Thus, we have $f(x^*, y) \in R^n_+$ for all $y \in K$, or equivalently, $x^* \in E^0$.

Conversely, if $x^* \in E^0$, then $x^* \in K$ and

$$f(x^*, y) \in R^n_+, \quad \forall y \in K.$$

It follows that for each $l \in I$, $f_l(x^*, y) \geq 0$ for all $y \in K$. Thus

$$\varphi_l(x^*) = \sup_{y \in K}\{-f_l(x^*, y)\} \leq 0$$

for all $l \in I$, which implies that

$$\varphi(x^*) = (-\varphi_1(x^*), \ldots, -\varphi_n(x^*))^T \in R^n_+. \tag{3.4}$$

Now, both (i) and (3.4) imply that $\varphi(x^*) = 0$.

From the proof of (ii), it is clear that $E^0 = \{x \in K : \varphi(x) = 0\}$ holds and thus the proof is complete. □

4 Applications to optimization problems with a system of vector equilibrium constraints

In this section, we consider a class of optimization problems with a system of vector equilibrium constraints (in short, OP). We first transform equivalently (OP) into a class of optimization problems with equality constraints by using generalized gap function of (SVEP) obtained in Sect. 3. Then, we give the dual problem of (OP), prove the concavity of the dual function, weak duality of (OP) and the saddle point sufficient condition under certain assumptions.

We consider (SVEP) in the case of $X_i = Y_i = R$ for all $i \in I$. It is clear that $X = \prod_{i \in I} X_i = Y = \prod_{i \in I} Y_i = R^n$. Assume that the following assumptions hold:

- (i) for each $i \in I$ and any $x \in K$, $\xi_{e_i}(x, f_i(x, x_i)) = 0$, where x_i is the ith component of x;
- (ii) for each $i \in I$ and any $x \in K$, the vector-valued function $y_i \mapsto f_i(x, y_i)$ is C_i-convex;
- (iii) for each $i \in I$ and any $y_i \in K_i$, the vector-valued function $x \mapsto f_i(x, y_i)$ is continuous;
- (iv) for each $i \in I$, the point-to-set mapping $W_i : K \to 2^{Y_i}$ has closed graph in $K \times Y_i$, where $W_i(x) = Y_i \backslash (-\text{int}C_i(x)), \forall x \in K$;
- (v) there exists a nonempty compact subset $D \subseteq K$ and for each $i \in I$, there exists a nonempty compact and convex subset $E_i \subseteq K_i$, such that $\forall x \in K \backslash D$, $\exists i \in I$ and $\exists \overline{y_i} \in E_i$ such that

$$\xi_{e_i}(x, f_i(x, \overline{y_i})) < 0.$$

Remark that from Lemma 2.1 (v), assumption (i) is equivalent to the condition that for any $x \in K$ and each $i \in I$, $f_i(x, x_i) \in -\partial C_i(x)$, where x_i is the ith component of x. Since assumptions (i)–(v) hold, E^S is nonempty and compact (see Theorem 4.1 of [23]), where $E^S = \{x^* \in K : \text{for each } i \in I : f_i(x^*, y_i) \notin -\text{int}C_i(x^*), \forall y_i \in K_i\}$.

Let $g : K \to R$ be a continuous function. We consider the following optimization problem with system of vector equilibrium constraints (in short, OP):

$$\min_{x \in E^S} g(x),$$

where E^S is called the feasible set of (OP). We denote by E^P the solutions set of (OP). Obviously, $E^P \subseteq E^S$. Since g is continuous and E^S is nonempty and compact, it is clear that (OP) is solvable, or equivalently, $E^P \neq \emptyset$. From Theorem 3.4, we have that $E^S = \{x \in K : \varphi(x) = 0\}$, where the function $\varphi : K \to R^n$ is defined by

$$\varphi(x) = (-\varphi_1(x), \ldots, -\varphi_n(x))^T$$

and

$$\varphi_i(x) = \sup_{y_i \in K_i} \{-\xi_{e_i}(x, f_i(x, y_i))\}, \quad i = 1, \ldots, n,$$

where the superscript T denotes the transpose. Thus, (OP) can be equivalently rewritten as the following optimization problem with equality constraints:

$$\min_{x \in K} g(x)$$

$$\text{subject to} \quad \varphi_i(x) = 0, \quad i = 1, \ldots, n.$$

Definition 4.1 The Lagrangian of (OP) is defined as

$$L(x, \lambda) = g(x) + \lambda^T \varphi(x), \quad \forall (x, \lambda) \in K \times R^n,$$

where $\lambda \in R^n$ is called the Lagrangian Multiplier of (OP).

Since

$$\sup_{\lambda \in R^n} L(x, \lambda) = \sup_{\lambda \in R^n} \{g(x) + \lambda^T \varphi(x)\}$$
$$= \begin{cases} g(x), & \text{if} \quad \varphi(x) = 0 \\ +\infty, & \text{otherwise}, \end{cases}$$

(OP) can be therefore restated in the form

$$\min_{x \in K} \max_{\lambda \in R^n} L(x, \lambda).$$

For $\lambda \in R^n$, define the function

$$d(\lambda) = \min_{x \in K} L(x, \lambda),$$

which is called the dual function. Then the dual problem (in short, DP) of (OP) is defined to the following unconstrained optimization problem:

$$\max_{\lambda \in R^n} d(\lambda).$$

We denote by E^D the solutions set of (DP). In order to derive the concavity of the dual function, we further assume that

(vi) for each $i \in I$, $e_i : K \to Y_i$ is continuous, $C_i : K \to 2^{Y_i}$ and $H_i : K \to 2^{Y_i}$ are upper semi-continuous, where $H_i(x) = Y_i \backslash \text{int} C_i(x)$ for all $x \in K$;
(v) for each $i \in I$ and $x \in K$, the function $y_i \mapsto \xi_{e_i}(x, f_i(x, y_i))$ is bounded from above.

Note that the pointed cone $C_i(x)$ implies that the properness (i.e., $C_i(x) \neq Y_i$). Thus, we have that for each $i \in I$, $(u, v) \mapsto \xi_{e_i}(u, v)$ is continuous on $K \times Y_i$ (see Theorem 2.1 of [10]). Consider assumption (iii) and the definition of φ_i, it follows that φ_i is continuous for all $i \in I$, so is φ. According the continuity of φ, we obtain the concavity of the dual function d, which is independent of the convexity of g and φ.

Theorem 4.1 (*Concavity of the dual function*) *If assumptions* (*i*)–(*vi*) *hold and K is compact, then the dual function d is concave.*

Proof Remark that g and φ are continuous and K is compact. Let $t \in [0, 1]$, then for any $\lambda_1, \lambda_2 \in R^n$,

$$\begin{aligned} d(t\lambda_1 + (1-t)\lambda_2) &= \min_{x\in K} L(x, t\lambda_1 + (1-t)\lambda_2) \\ &= \min_{x\in K}\{g(x) + (t\lambda_1 + (1-t)\lambda_2)^T \varphi(x)\} \\ &= \min_{x\in K}\{t[g(x) + (\lambda_1)^T\varphi(x)] + (1-t)[g(x) + (\lambda_2)^T\varphi(x)]\} \\ &\geq t\min_{x\in K}\{g(x) + (\lambda_1)^T\varphi(x)\} + (1-t)\min_{x\in K}\{g(x) + (\lambda_2)^T\varphi(x)\} \\ &= t\min_{x\in K} L(x,\lambda_1) + (1-t)\min_{x\in K} L(x,\lambda_2) \\ &= td(\lambda_1) + (1-t)d(\lambda_2). \end{aligned}$$

This proof is complete. □

Theorem 4.2 (*Weak Duality*) *If $x^* \in E^P$, then for any feasible point $x \in E^S$ and $\lambda \in R^n$,*

$$d(\lambda) \leq d(\lambda^*) \leq g(x^*) \leq g(x),$$

for all $\lambda^* \in E^D$.

Proof Since $\varphi(x^*) = 0$, it follows that for any $\lambda \in R^n$, $\lambda^T\varphi(x^*) = 0$. Consider the definition of d, one has $d(\lambda) \leq g(x^*) + \lambda^T\varphi(x^*)$ for all $\lambda \in R^n$. Thus for any all feasible point $x \in E^S$, $d(\lambda) \leq \max_{\lambda\in R^n} d(\lambda) = d(\lambda^*) \leq g(x^*) = \min_{x\in E^G} g(x) \leq g(x)$, for all $\lambda^* \in E^D$. This proof is complete.

As is well know, if $d(\lambda^*) < g(x^*)$, then the difference $g(x^*) - d(\lambda^*)$ is called the "duality gap". The following conclusion is an immediate consequence of Theorem 4.2. □

Corollary 4.1 *If $x^* \in E^S$ and $\lambda^* \in R^n$ such that*

$$d(\lambda^*) = g(x^*),$$

then $x^* \in E^P$ and $\lambda^* \in E^D$.

Definition 4.2 (x^*, λ^*) is called a saddle point of the Lagrangian L if, $x^* \in K$ and $\lambda^* \in R^n$ such that

$$L(x^*, \lambda) \leq L(x^*, \lambda^*) \leq L(x, \lambda^*), \quad \forall x \in K,\ \lambda \in R^n. \tag{4.1}$$

Based on the Lagrangian of (OP), we have the following conclusion.

Theorem 4.3 (*Saddle point sufficient condition*) *If (x^*, λ^*) is a saddle point of the Lagrangian L, then $x^* \in E^P$.*

Proof Since (x^*, λ^*) is a Saddle point of the Lagrangian L, the first inequality of (4.1) implies that

$$(\lambda - \lambda^*)^T \varphi(x^*) \leq 0, \quad \forall \lambda \in R^n. \tag{4.2}$$

We conclude that $\varphi_j(x^*) = 0$ for all $i = 1, \ldots, n$. Suppose that there exists j_0 such that $\varphi_{j_0}(x^*) \neq 0$. Then, we choose λ_{j_0} such that $\lambda_{j_0} > \lambda^*_{j_0}$, if $-\varphi_{j_0}(x^*) > 0$, and $\lambda_{j_0} < \lambda^*_{j_0}$, if $-\varphi_{j_0}(x^*) < 0$. Thus, $\lambda = (\lambda^*_1, \ldots, \lambda^*_{j_0-1}, \lambda_{j_0}, \lambda^*_{j_0+1} \ldots, \lambda^*_n)^T \in R^n$ leads to a contradiction with (4.2). It follows that $x^* \in K$ is feasible, that is, $x^* \in E^S = \{x \in K : \varphi(x) = 0\}$. The second inequality of (4.1) and the feasibility of x^* imply that

$$g(x^*) \leq g(x) + (\lambda^*)^T(\varphi(x) - \varphi(x^*)) = g(x) + (\lambda^*)^T \varphi(x) = g(x)$$

for all $x \in E^S = \{x \in K : \varphi(x) = 0\}$, which implies that $x^* \in E^P$. The proof is complete.
From Corollary 4.1 and Theorem 4.3, we obtain the following conclusion. □

Corollary 4.2 *If* (x^*, λ^*) *is a saddle point of the Lagrangian L such that*

$$d(\lambda^*) = g(x^*),$$

then $x^* \in E^P$ *and* $\lambda^* \in E^D$.

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